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N = 4 Supersymmetric (Dyonic) Hypermultiplets in String Theory

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In four-dimensional $N = 4$ supersymmetric gauge theory, we obtain an exact metric on the moduli space of quantum vacua and analyze the spectra of BPS states in weak as well as in strong coupling regions. Identifying the hypermultiplet of the dyonic state as a string stretched between D_3-brane probe and a 7-brane, we demonstrate that the two hypermultiplets, which become massless at two singularities in supersymmetric theory, correspond to open strings beginning on the D_3-brane and ending on the respective 7-brane.

KEY WORDS: supersymmetry; electric magnetic duality; monodromies; brane; moduli space; open-string.

1. INTRODUCTION

Recently, some major progress has been obtained in the understanding of dynamics of $N = 1$ supersymmetric gauge theories of monopoles and dyons in four dimensions (Rajput *et al.*, 1991; Seiberg, 1994) and some important results have been obtained (Gukov and Polyubin, 1997; Terashima and Yang, 1997; Witten and Seiberg, 1994) about their strong coupling by using holomorphic properties of the superpotential and gauge kinetic function (Intriligator, 1994; Intriligator and Seiberg, 1994; Seiberg *et al.*, 1994) culminating in Seiberg's nonabelian duality conjecture (Argyres *et al.*, 1996; Seiberg, 1995). Following this work, huge progress has been made during last couple of years in understanding the four-dimensional $N = 2$ supersymmetric gauge theories. Recently, we have undertaken (Joshi *et al.*, 2000; Singh and Rajput, 1999a) the study of monopoles and dyons in four-dimensional $N = 2$ and in $N = 4$ supersymmetric theories and obtained

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(Joshi *et al.*, 2001; Singh and Rajput, 1999b) complete BPS spectra by analyzing the kinematics and monodromies around singularities in quantum moduli space without as well as with hypermultiplets. The ability to make exact statements (Bilal and Ferrari, 1996) in these four-dimensional strongly coupled field theories makes them interesting in the laboratories where various ideas about quantum field theory can be tested and the existence of monopoles and dyons possibly verified.

The interplay between the space time symmetries and world sheet symmetries for the BPS states as first studied (Callan *et al.*, 1991) in connection with the heteriotic instantons and solitons and its analysis as performed (Gauntlett *et al.*, 1993; Khuri, 1992) for the five-branes and the related H-monopoles. Properties of electrically charged supersymmetric solutions have been compared with the properties of the states in string theories (Sen, 1995) and many of the electrically charged solutions have been interpretted as string states (Duff and Rahmfeld, 1995). In string theory, $N = 2$ SU(2) supersymmetry has been analyzed (Bank *et al.*, 1996; Bergman and Fayyazuddin, 1998) as the low energy theory on a D_3-brane probe in the background of an orientified 7-plane (Ω^7).

This paper is organized as follows. In section 2, we have analyzed the structure of scale invariant $N = 4$ theory in the absence of hypermultiplets and interpret the singularities. In section 3, we have carried out the study of moduli space vacua in four-dimensional $N = 4$ supersymmetric theory with gauge group SU(2) and analyzed its kinematics at classical as well as quantum levels. Analyzing the monodromies around singularities in quantum moduli space, the spectrum of BPS states of dyons have been obtained in weak coupling region R_W and the strong coupling regions R_S and it has been shown that in the absence of hypermultiplets (i.e., quarks) the weak coupling spectrum contains all dyons $(n, 1)$ while the strong coupling spectrum consists of monopoles $(0, 1)$ and dyons $(1, \pm 1)$. It has been demonstrated that, in the presence of hypermultiplets, the duality in the theory with nonzero bare masses is really an electric–magnetic-flavor duality. It is shown that for $N_f = 1$ (single flavor) the theory exhibits, Z_3-symmetry, and the massless states $(0, 1)$, $(-1, 1)$, $(1, 1)$, and $(-2, 1)$ exist both in the weak and strong coupling regions. It has been shown that the strong coupling region, for $N_f = 2$ case, consists of only one Z_2-pair incorporating monopoles $\pm(0, 1)$ and dyons $\pm(\pm 1, -1)$ while that for $N_f = 3$ case it consists of dyons $\pm(-1, 2)$, $\pm(1, -1)$ and monopoles $\pm(0, 1)$. Identifying a (n_e, n_m) hypermultiplet of dyonic state in the probe theory as a (n_e, n_m) string stretched between the D_3-brane probe and a 7-brane, it has been demonstrated that two hypermultiplets, which become massless at the two singularities in $N = 4$ SU(2) supersymmetric theory, correspond to open strings beginning on the D_3-brane and ending on the respective 7-brane. It has also been shown that other than these two multiplets, all the BPS states, including W-bosons, correspond to multipronged strings connecting the D_3-brane with the two 7-branes.

2. CURVE FOR SCALE INVARIANT N = 4 SUPERSYMMETRIC THEORY

In the absence of bare masses, this theory is conformally invariant. This theory has a dimensionless coupling constant (Witten and Seiberg, 1994),

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi\, i}{g^2}. \tag{2.1}$$

Therefore, in the curve $y^2 = F(x, u, m_i, \tau)$ that controls the low energy behavior, the coefficients are functions of τ that must be determined. This contrasts with $N_\mathrm{f} < 4$ where, instead of τ, one has the renormalization scale Λ; dimensional analysis ensures that the dependence on Λ is polynomial, so that there are only finitely many parameters to determine.

We assume the massless case for finding the right family of curves for the conformally invariant theory, that is when the bare mass is zero. In this case classical formula

$$\begin{aligned} a(u) &= \sqrt{2}u, \\ a_\mathrm{D}(u) &= \tau a = \left(\frac{\theta}{2\pi} + \frac{4\pi\pi}{g^2}\right)a, \end{aligned} \tag{2.2}$$

is exact. So we wish to find a curve

$$y^2 = F(x, u, \tau),$$

such that the differential form

$$\omega = \frac{\sqrt{2}}{4\pi}\,\frac{dx}{y}, \tag{2.3}$$

has the periods $(\frac{\partial a_\mathrm{D}(u)}{\partial u}, \frac{\partial a(u)}{\partial u})$, with $(a_\mathrm{D}(u), a(u))$ given by Eq. (2.2).
Now, a genus one curve E and a differential form with periods a multiple of $(\tau, 1)$ can be found as follows. Let E be the quotient of the complex z-plane by the lattice generated by π and $\pi\tau$. Let $\omega_0 = dz$. Obviously, the periods of ω_0 are π and $\pi\tau$, integrated on contours that run from 0 to π and from 0 to $\pi\tau$.

For analyzing an algebraic description on E, one introduces the Wieerstrass $\wp$ function, which obeys

$$\wp(z) = \wp(z+1) = \wp(z+\pi) = \wp(-z), \tag{2.4}$$

and has for its only singularity on E a double pole at the origin. If one sets $x_0 = \wp(z)$, $y_0 = \wp'(z)$, then one finds

$$y_0^2 = 4x_0^3 - g_2(\tau x_0) - g_3(\tau), \tag{2.5}$$

where

$$\begin{aligned} g_2 &= 60\pi^{-4}G_4(\pi), \\ g_3 &= 140\pi^{-6}G_6(\pi), \end{aligned}$$

and G_4, G_6 are the usual Einstein series

$$G_4(\pi) = \sum_{m,n \in z_{\neq 0}} \frac{1}{(m\tau + n)^4},$$
$$G_6(\pi) = \sum_{m,n \in z_{\neq 0}} \frac{1}{(m\tau + n)^6}, \tag{2.6}$$

which defines modular forms for SL (2, Z) of weight 4 and 6, respectively. Since the definition of x_0 and y_0 was such that $y_0 = \frac{dx_0}{dz}$, one can also rewrite

$$\omega_0 = dz \quad \text{as} \quad \omega_0 = \frac{dx_0}{y_0}. \tag{2.7}$$

Now, set

$$x = x_0 u \qquad y = \frac{1}{2} y_0 u^{3/2},$$

and (2.8)

$$\omega = \frac{\sqrt{2/u}}{2\pi}\omega_0 = \frac{\sqrt{2}}{4\pi}\frac{dx}{y}.$$

The equation for the curve becomes

$$y^2 = x^3 - \frac{1}{4}g_2(\tau)xu^2 - \frac{1}{4}g_3(\tau)u^2. \tag{2.9}$$

This change of variables and in particular the normalization of u is motivated by the following requirement. For weak coupling ($\tau \to i\infty$) we should recover our curve $y^2 = F_0(x, u) = x^2(x - u)$. It is easy to check from the asymptotic behavior $g_2 = \frac{4}{3} + O(q), g_3 = \frac{8}{27} + O(q)$ that after replacing x in Eq. (2.9) by $x - 4/3$ we find F_0.

The period of ω is now $\frac{1}{2}\sqrt{2/u(1, \tau)}$ for $N = 4$ and, one has

$$a(u) = \sqrt{2u},$$
$$a_{\mathrm{D}}(u) = \tau a, \tag{2.10}$$

as desired. We have determined the curve for the massless theory. The coefficients are in modular forms. This test is equivalent to the fact that the metric of the classical theory is S-dual, which is one of the traditional pieces of evidence for S-duality.

We recall that $N = 4$ symmetric Yang–Mills can be regarded as $N = 2$ super Yang–Mills theory with an additional matter field that is a hypermultiplet in the adjoint representation of the gauge group. One can give a bare mass m—to that hypermultiplet, explicitly breaking $N = 4$ to $N = 2$. We analyze the theory for gauge group SU(2). For weak coupling region $|q| \ll 1$, with $m \neq 0$. There is one singularity at $u \approx \frac{1}{4}m^2$ where a component H of the elementary hypermultiplet is massless. This gives a monodromy conjugate to T^2. Since the elementary hypermultiplet for $N = 4$ has twice the electric charge of the hyperdoublets. As a hyperdoublet gives monodromy T, and the one loop beta function, which determines the monodromy is proportional to the square of the charge, the massless H particle would give monodromy T^2 in the conventions of $N = 4$. In addition, at an energy of order $\Lambda_0 \sim q^{1/4}m$, the theory evolves to a strongly coupled pure $N = 2$ gauge theory. This theory has two singularities, associated with massless monopoles, with monodromies conjugate to T^2. So altogether, we get three singularities, each conjugate to T^2. These three singularities are permuted under monodromies in q and m. This is a reason the $N = 4$ conventions in which they are all conjugate to T^2 are preferable to the $N_\mathrm{f} \neq 0$ conventions in which one is conjugate T^4 and the other to T.

This analysis is valid for very weak coupling. Could it be, for instance, that what we described above as one conjugate to T, separated by an amount that vanishes for weak coupling. SL (2, Z) group theory alone would permit this, but it is impossible because each of the singularities arises when a single hypermultiplet becomes massless. So we are looking for a family of curves

$$Y^2 = F(x, u), \tag{2.11}$$

(with cubic F) that as u varies has precisely three singularities each conjugate to Y^2. There is a singularity at u_0 with monodromy T^n for $n > 1$ (will be 2) if and only if for some x_0,

$$F = \frac{\partial F}{\partial a} = \frac{\partial F}{\partial u} = 0, \tag{2.12}$$

at $x = x_0$, $u = u_0$. Eq. (2.12) means that the curve $F(x, y) = 0$ has a singularity at $(x, u) = (x_0, u_0)$.

We analyze a plane cubic curve $F(x, u) = 0$ with three distinct singularities. The possible singularities of a plane cubic curve can be completely classified. If F is an irreducible polynomial, there is at most one singularity. If $F_1 F_2$, with F_1 linear in x and u and F_2 quadratic and irreducible, there are precisely two singularities (perhaps at infinity), namely the points where $F_1 = F_1 = 0$. The only way to get three singularities is to have $F = F_1 F_2 F_3$, where the three factors are linear; the three singularities are the points $F_i = F_j = 0$ for any two distinct i and j. For reproducing the known $m = 0$ limit of F, the F_i must be (up to a scalar multiple and a permutation of i) $F_i = x - e_i u - F_i$, where F_i are the functions of m and τ only and vanishes at $m = 0$ and e_i being the roots of the cubic polynomial (2.5) obey $e_1 + e_2 + e_3 = 0$.

Moreover, by adding constants (that is, functions of m and τ only) to x and u, one can eliminate two of the F_i. Since we did not assign any physical meaning to x we can take the freedom to shift it. However, we want to preserve

$$u = \langle T_r \varphi^2 \rangle. \tag{2.12a}$$

Therefore, we will denote the shifted u by $\tilde{u}$. To keep symmetry under permuting the e_i, we shift x and u such that $f_i = e_i^2 f$. Then the equation of mass-deformed $N = 4$ theory becomes

$$y^2 = \left(x - e_1\tilde{u} - e_1^2 f\right)\left(x - e_2\tilde{u} - e_2^2 f\right)\left(x - e_3\tilde{u} - e_3^3 f\right), \tag{2.13}$$

the relation between u and $\tilde{u}$ is determined by examining the theory at weak coupling; i.e., in the limit $\tau \to i\infty$. In this limit we should reproduce on weak coupling curve $y^2 = E_0 = x^2(x - u)$. This motivates us to change variables to

$$\begin{aligned} \tilde{u} &= u - \frac{1}{2}e_i f \\ x &= x - \frac{1}{2}e_i u + \frac{1}{2}e_1^2 f, \end{aligned} \tag{2.14}$$

in Eq. (2.13). The family of curves becomes

$$y^2 = (x - c_1 u)(x + c_2 u - c_1(c_1 + c_2) f)(x - c_2 u + c_2(c_1 + c_2) f), \tag{2.15}$$

with

$$c_1 = \frac{3}{2}e_1 \quad \text{and} \quad c_2 = \frac{1}{2}(e_3 - e_2).$$

In this form it is easy to study the weak coupling limit. For a smooth limit, $f_0 = f(\tau = i\infty)$ should be finite. Using $c_1(\tau = i\infty) = 1$ *and* $c_2(\tau = i\infty) = 0$, the exact curve (2.15)

$$y^2 = F_0 = x^2(x - u),$$

as required. And so in the form (2.15) the family of curves is expressed in terms of Eq. (2.12a).

We can relate $f_0 = f(\tau = i\infty)$ to the mass by examining the singularities of (2.15). The roots of the equation are at $x_1 = c_1^4$ and $x_{2,3} = \pm c_2(-u + (c_1 \pm c_2) f)$. A singularity occurs when $x_i = x_j$. This occurs for

$$\begin{aligned} u_1 &= \frac{3}{2}e_i f = c_1 f, \\ u_{2,3} &= \pm\frac{1}{2}(e_3 - e_2) f = \pm c_2 f. \end{aligned} \tag{2.16}$$

In the weak coupling limit $c_1 \approx 1$, $c_2 \approx 8q^{1/2}$ and have $u_1 \approx f_0$ *and* $u_{2,3} \approx \pm 8q^{1/2} f_0$. We interpret the singularity at u_1 as associated with the massless elementary field. It should be at $a = m/\sqrt{2}$. For weak coupling and in the $N = 4$ normalization this is at $u \approx \frac{1}{2}a^2 = m^2/4$.

Therefore

$$f_0 = f(\tau = i\infty) = \frac{m^2}{4}. \tag{2.17}$$

The other two singularities at $u_{1,2} \approx \pm 8q^{1/2} f_0$ are interpreted as the two singularities of the low energy pure gauge $N = 2$ theory. More precisely, we can now take the scaling limit $q \to 0,\ f_0 = \frac{m^2}{4} \to \infty$ holding $\Lambda^2 = 2q^{1/2}m^2$ fixed. In this limit, Eq. (2.15) becomes

$$y^2 = (x-4)(x^2 - \Lambda^4), \tag{2.18}$$

which is exactly the curve of the expected low energy pure gauge $N = 2$ theory in the $N = 4$ conventions with scale Λ. The curve (2.13) governing the low energy behavior of the mass-deformed $N = 4$ theory can be written as

$$y^2 = \left(x - e_1\tilde{u} - \frac{1}{4}e_1^2m^2\right)\left(x - e_2\tilde{u} - \frac{1}{4}e_2^2m^2\right)\left(x - e_3\tilde{u} - \frac{1}{4}e_3^2m^2\right), \tag{2.19}$$

with

$$f = \frac{1}{4}m^2, \qquad \text{and} \qquad u = \langle Tr\varphi^2\rangle = \tilde{u} + \frac{1}{8}e_1m^2. \tag{2.20}$$

This formula is completely SL (2, Z) invariant; the coefficients are modular forms. Since the formula is not limited to weak coupling, this SL (2, Z) invariance is a genuine, new, strong coupling test of SL (2, Z) invariance of the $N = 4$ theory.

3. QUANTUM MODULI SPACE AND MONODROMIES AROUND THE SINGULARITIES IN N = 4 SUPERSYMMETRIC THEROY

The classical potential of $N = 4$ supersymmetric theory in the absence of hypermultiplets $N_\mathrm{f} = 0$ is

$$V(\phi) = \frac{1}{|g|^2}\mathrm{tr}\{(\phi_i, {\phi_i}^+)(\phi_j, {\phi_j}^+)\}, \tag{3.1}$$

where ϕ is the Higg's field and g is the gauge coupling constant of the underlying microscopic theory. As long as ϕ and ϕ^+ commute, this potential vanishes even for the nonvanishing expectation value of ϕ;

$$\langle 0|\phi|0\rangle = a \neq 0.$$

It shows that the theory has a continuum of gauge in-equivalent vacua called the classical moduli space parameterized by

$$u = \mathrm{tr}\phi_i\phi_j = \frac{1}{2}a^2, \tag{3.2}$$

where for SU(2) gauge group, we have set

$$\phi = \frac{1}{2}a\sigma_3,$$

with

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For generic ϕ, the low energy effective Lagrangian containing a single $N = 4$ vector multiplet may be expressed in terms of holomorphic function

$$F(a) = \frac{1}{2}\tau_{\text{cl}} a^2, \tag{3.3}$$

where τ_{cl}, the classically effective coupling constant in the vacuum parameterized by a, is defined as

$$\tau_{\text{cl}} = \frac{8\pi i}{g^2} = \frac{\partial^2 F}{\partial a^2}, \tag{3.4}$$

The dyonic charge and mass for BPS states may then be written in the following forms (Singh and Rajput, 1999b) respectively;

$$q = (n_{\text{e}}, n_{\text{m}}) = \sqrt{2}\left(n_{\text{e}} - \frac{8\pi i}{g^2} n_{\text{m}}\right) = \sqrt{2}(n_{\text{e}} - \tau_{\text{cl}} n_{\text{m}}), \tag{3.5}$$

and

$$\text{M} = a|q| = a\sqrt{2}|n_{\text{e}} - \tau_{\text{cl}} n_{\text{m}}|, \tag{3.6}$$

where n_{e} and n_{m} are electric charge and magnetic charge numbers respectively.

In coulomb phase, the gauge theory has massless photon and hence it is subjected to standard electric–magnetic duality

$$q = (n_{\text{e}}, n_{\text{m}}) \to (-n_{\text{m}}, n_{\text{e}}) = -\frac{1}{\tau_{\text{cl}}} q = q', \tag{3.7}$$

which incorporates the inversion of τ_{cl}. On the other hand under the duality transformation

$$\tau_{\text{cl}} \to \tau_{\text{cl}} + 1 \tag{3.8}$$

we have

$$q = (n_{\text{e}}, n_{\text{m}}) \to (n_{\text{e}} - n_{\text{m}}, n_{\text{m}}) = q'. \tag{3.9}$$

The transformations (3.7) and (3.8) generate an infinite duality group SL(2, Z).

The quantum theory also has a vacuum moduli space, the metric on which is Kähler metric which may be written locally as

$$ds^2 = -\frac{i}{2}(da_{\text{D}} d\bar{a} - da\, d\bar{a}_{\text{D}}), \tag{3.10}$$

where

$$a_{\text{D}} = \frac{\partial F}{\partial a}. \tag{3.11}$$

Then Eq. (3.6), for mass of BPS state, may be written as

$$M = \sqrt{2}|Z|, \tag{3.12}$$

where

$$Z = an_{\mathrm{e}} - a_{\mathrm{D}}n_{\mathrm{m}}, \tag{3.12a}$$

is the central charge of the supersymmetric algebra. Here a is related by $N = 4$ theory to the semiclassical photon while a_{D} is related to its dual "the magnetic photon." The existence of a BPS state $(n_{\mathrm{e}}, n_{\mathrm{m}})$ at u in the moduli space implies the existence of another BPS state (n'_e, n'_m) at u' when u and u' are related by a global symmetry acting on the moduli space, and the electric and magnetic charge numbers are related as

$$\begin{pmatrix} n'_{\mathrm{e}} \\ n'_m \end{pmatrix} = G\begin{pmatrix} n_{\mathrm{e}} \\ n_{\mathrm{m}} \end{pmatrix}, \tag{3.13}$$

where matrix $G \in \mathrm{SL}(2, Z)$, generated by transformations (3.7) and (3.8), and there exists a phase $e^{i\omega}$ such that

$$\begin{pmatrix} a_{\mathrm{D}}(u') \\ a(u') \end{pmatrix} = e^{i\omega}G\begin{pmatrix} a_{\mathrm{D}}(u) \\ a(u) \end{pmatrix}. \tag{3.14}$$

The explicit form of $a(u)$ and $a_{\mathrm{D}}(u)$ in terms of the period of a meromorphic differential of the second kind on a genus surface can be written from Eq. (2.18) as

$$y^2 = (x^2 - \Lambda^4)(x - u), \tag{3.15}$$

where Λ is the dynamically generated mass scale. This form gives a double covering of the plane branches at $\pm\Lambda^2$ and ∞. On these cuts the correctly normalized meromorphic one-form for this torus is

$$\lambda = \frac{\sqrt{2}}{2\pi}\int \frac{dx(x-u)}{y} = \frac{\Lambda\sqrt{2}}{2\pi}\int \frac{dx(x-u/\Lambda^2)^{1/2}}{\sqrt{(x^2-1)}},$$

which yields

$$a(u) = \frac{\Lambda\sqrt{2}}{\pi}\int_{-1}^{1} \frac{dt\sqrt{\frac{u}{\Lambda^2}-t}}{\sqrt{1-t^2}},$$

and

$$a_{\mathrm{D}}(u) = \frac{\Lambda\sqrt{2}}{\pi}\int_{1}^{u/\Lambda^2} dt\,\frac{\sqrt{\frac{u}{\Lambda^2}-t}}{\sqrt{1-t^2}}, \tag{3.16}$$

or

$$a(u) + a_{\mathrm{D}}(u) = \frac{\Lambda\sqrt{2}}{\pi}\int_{-1}^{u/\Lambda^2} dt\,\frac{\sqrt{\frac{u}{\Lambda^2}-t}}{\sqrt{1-t^2}}.$$

Representing these integrals in terms of the usual hyper-geometric function

$$F(\alpha, \beta, \gamma; x) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta-1} (1-xt)^{-\alpha}\, dt,$$

where $B(\beta, \gamma - \beta)$ is the usual beta function, we get (Singh and Rajput, 1999b)

$$\begin{aligned} a(u) &= \sqrt{\frac{u+\Lambda^2}{2}} F\left(-\frac{1}{2}, \frac{1}{2}, 1; \frac{2}{1+u/\Lambda^2}\right), \\ a_{\mathrm{D}}(u) &= i\sqrt{\frac{u-\Lambda^2}{2}} F\left(\frac{1}{2}, \frac{1}{2}, 2; \frac{1-u/\Lambda^2}{2}\right). \end{aligned} \tag{3.17}$$

Near the point at infinity the asymptotic behaviors of these functions for $\Lambda = 1$ are given by

$$\begin{aligned} a(u) &\sim \sqrt{\frac{u}{2}}, \\ a_{\mathrm{D}}(u) &\sim \frac{i}{\pi}\sqrt{2u}[\ln u + 3\ \ln 2 - 2]. \end{aligned} \tag{3.18}$$

Near the branch point $u = +1$ and $u = -1$ (for $\Lambda = 1$), the asymptotic behaviors of these functions are respectively given by the following sets of equations;

$$\begin{aligned} a(u) &\sim \frac{i}{2\pi}\left[(u+1)\ \ln\left(\frac{1+u}{2}\right) + \frac{u+1}{2}(-i\pi - 4\ \ln 2 + 3)\right], \\ a_{\mathrm{D}}(u) &\sim \frac{i}{\pi}\left[-\frac{1+u}{2}\ \ln\left(\frac{1+u}{2}\right) + \frac{u+1}{2}(1 + 4\ \ln 2) - 4\right], \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} a(u) &\sim \frac{2}{\pi} + \frac{1}{2\pi}\left(\frac{1-u}{2}\right)\left[\ln\frac{(u-1)}{2} + 1 - 4\ \ln 2\right], \\ a_{\mathrm{D}}(u) &\sim i\frac{u-1}{2}. \end{aligned} \tag{3.20}$$

From Eqs. (3.4) and (3.11) we have

$$\tau_{11} = \tau_{\mathrm{cl}} = \frac{\partial a_{\mathrm{D}}}{\partial a} = \frac{iF\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{u-\Lambda^2}{u+\Lambda^2}\right)}{F\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{2\Lambda^2}{u+\Lambda^2}\right)}, \tag{3.21}$$

which blows up at the cuts $u = \pm\Lambda^2$.

Let us invert $a = a(u)$ as $u = u(a)$ and then integrate Eq. (3.11) to get the holomorphic function $F(a)$. If a matrix $\Gamma \in \mathrm{SL}(2, Z)$ is taken as

$$\Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{3.22}$$

then

$$a'_{\mathrm{D}} = \alpha a_{\mathrm{D}} + \beta a, \qquad a' = \gamma \alpha_{\mathrm{D}} + \delta a, \tag{3.22a}$$

which gives the following transformation of the period matrix τ_{11};

$$\tau^1_{11} = \frac{\beta + \alpha\tau_{11}}{\delta + \gamma\tau_{11}} \tag{3.23}$$

These relations yield the following modular transformation of the homomorphic function $F(a)$;

$$F'(a) = F_\Gamma(a') = \frac{1}{2}\beta\delta a^2 + \frac{1}{2}\alpha\gamma a^2_{\mathrm{D}} + \beta\gamma\ a a_{\mathrm{D}} + F(a) \tag{3.24}$$

For the transformation matrix

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.22b}$$

corresponding to the transformation (3.7), Eqs. (3.22), (3.23), and (3.24) give

$$a'_{\mathrm{D}} = a^S_{\mathrm{D}} = a; \qquad a' = a^S = -a_{\mathrm{D}}, \tag{3.25}$$

$$\tau^1_{11} = \tau^S_{11} = -1/\tau_{11}, \tag{3.26}$$

and

$$F'(a) = F_S(a^S) = -a a_{\mathrm{D}} + F(a). \tag{3.27}$$

Similarly, for the transformation matrix

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{3.22c}$$

corresponding to (3.9), we have

$$a'_{\mathrm{D}} = a^T_{\mathrm{D}} = a_{\mathrm{D}} + a; \qquad a' = a^T = a, \tag{3.28}$$

$$\tau^1_{11} = \tau^T_{11} = 1 + \tau_{11}, \tag{3.29}$$

$$F'(a) = F_T(a^T) = \frac{1}{2}a_2 + F(a). \tag{3.30}$$

The transformation (3.7) incorporates the transformation of an electric charge (1, 0) to a monopole (0, 1), i.e., it leads to the monopole-region. On the other hand, the transformation (3.9) incorporates the transformation of a monopole (0, 1) to a dyon (−1, 1), i.e., it leads to the dyon-region. Equations (3.25) and (3.28) show that in the monopole and dyon regions the natural independent variables to be used are

$$a^{(m)} = a_{\mathrm{D}}, \quad \text{and} \quad a^{(d)} = a_{\mathrm{D}} - a,$$

respectively, with the corresponding prepotentials $F^{(m)}_{(a^m)}$ and $F^{(d)}_{(a^d)}$ given by Eqs. (3.27) and (3.30) respectively.

Monodromy is a transformation of the parameter $u = \frac{1}{2}a^2$ of the quantum moduli space to the point u_0 where the mass of BPS state, as given by Eq. (3.12), vanishes. Thus the values of $(n_{\mathrm{e}}, n_{\mathrm{m}})$ of the massless particle at singularity are determined by the monodromy around the singularity, i.e.,

$$n_{\mathrm{e}} a - n_{\mathrm{m}} a_{\mathrm{D}} = 0. \tag{3.31}$$

The moduli space M may be divided in two region R_{S} and R_{W} by a curve C of marginal stability, i.e.,

$$C = \left\{ u \in M \Big/ \mathrm{Im}\left(\frac{a_{\mathrm{D}}}{a}\right) = 0 \right\}, \tag{3.32}$$

The region inside C is the strong coupling region R_{S} while the region out side C is the weak coupling region R_{W}. If two point u and u' in M can be joined by a continuous path in M without crossing C, then one can transform u into u' without changing the spectrum. For instance, if $u' = u_0$ (where the mass of BPS state vanishes), then we have

$$\begin{pmatrix} a'_{\mathrm{D}} \\ a' \end{pmatrix} = M(n_{\mathrm{e}}, n_{\mathrm{m}}) \begin{pmatrix} a_{\mathrm{D}} \\ a \end{pmatrix}, \tag{3.33}$$

where

$$M(n_{\mathrm{e}}, n_{\mathrm{m}}) = \begin{pmatrix} 1 + 2n_{\mathrm{e}}n_{\mathrm{m}} & 2n_e^2 \\ -2n_{\mathrm{m}}^2 & 1 - 2n_{\mathrm{e}}n_{\mathrm{m}} \end{pmatrix}, \tag{3.34}$$

without changing the spectrum. The curve C of Eq. (3.32) looks like an ellipse. The massless BPS states can exist only on this curve and hence the monodromy transformation can be constructed only on this curve. At $u = \Lambda^2 = 1$ the monopole is massless and at $u = -\Lambda^2 = -1$ the massless state is dyonic described as $\pm(1, -1)$ if we approach $u = -1$ from upper half plane or as $\pm(1, 1)$ if the point $u = -1$ is approached from the lower half plane. As such the monopole $\pm(0, 1)$ and the dyon either as $\pm(1, 1)$ or as $\pm(1, -1)$ do exist in both R_{W} and R_{S} and there is no other state in R_{W} and R_{S} that becomes massless on C since there are precisely two singularities, $u = 1$ and $u = -1$, where

$$\frac{a_{\mathrm{D}}}{a} = \pm 1 \text{ or } 0.$$

At the singularity $u = 1$, the asympotic relations (3.19) gives **Au: OK?**

$$a_D \to a'_D = a_D,$$

and

$$a \to a' = a - 2a_D. \tag{3.35}$$

Comparing these transformations with that given by Eq. (3.33), we get the following monodromy matrix at $u = 1$

$$M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}. \tag{3.35a}$$

This monodromy arises from a massless monopole (0, 1). Under this monodromy we have the following transformation of charges:

$$q = (n_e, n_m) \to (n_e, n_m + 2n_e) = q'. \tag{3.35b}$$

Similarly, by using relations (3.20), we get the monodromy matrix

$$M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \tag{3.36}$$

at $u = -1$. Under this monodromy

$$q = (n_e, n_m) \to (2n_m - n_e, 3n_m + 2n_e) = q'. \tag{3.36a}$$

This monodromy arises from the vanishing mass of dyon (1, −1). The monodromy matrix at $u = \infty$ is

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \tag{3.37}$$

which incorporates the following transformation of charge

$$q = (n_e, n_m) \to (-n_e - 2n_m, -n_m), \tag{3.38}$$

showing that at infinity a monopole (0, 1) gains the electric charge and becomes a dyon (−2, −1). Monodromy at infinity implies that there must be an additional singularity somewhere in u-plane.

From Eqs. (3.35), (3.36), and (3.37), we obviously have

$$M_1 M_{-1} = M_\infty. \tag{3.39}$$

These matrices form the subgroup $\Gamma(2) \in SL(2, Z)$. In looping around $u = 1$ and $u = -1$, the pair (a_D, a) are transformed by monodromies M_1 and M_{-1} and the charges (n_e, n_m) are transformed accordingly. But the spectrum of BPS saturated states is not $\Gamma(2)$ invariant. This lack of duality in the spectrum can be resolved if the curve, on which $a_{D/a}$ is real, looks some thing like $|u| = 1$. Then one can avoid the jumping phenomena only if one stays in the region $u > 1$. The only monodromy that can be seen in that region is M_∞, under which the spectrum of BPS saturated states is invariant. Whatever particles becomes massless at $u = \pm 1$ must evolve continuously from the BPS-saturated states that can be seen semiclassically near infinity.

The strong coupling region R_S contains exactly two BPS states, the monopole (0, 1) and dyon (1, 1) or (1, −1) that becomes massless at $u = \Lambda^2$ or $u = -\Lambda^2$ while all dyons $(n, 1)$, where n is in integer, exist in the weak coupling region R_W (Singh and Rajput, 1999b).

All the results of this section for $N_f = 0$ can be generalized to the general $N = 4$ supersymmetric Yang Mills theories with gauge group SU(2) when the hypermultiplets are present. In the presence of N_f flavors with m_f as the bare mass of hypermultiplets $(Q_f, \bar{Q}_f)$, the Eq. (3.12a) for quantized central charge may be generalized in the following form

$$Z = an_e - a_D n_m + \frac{1}{\sqrt{2}} \sum_{f=1}^{N_f} m_f s_f, \tag{3.40}$$

where s_f is the quark number charge. In this case we have a singular point when (Singh and Rajput, 1999b)

$$a = \pm \frac{m_f}{\sqrt{2}} = a_0,$$

which corresponds to quark becoming massless. The monodromy transformation around this singularity is

$$\begin{aligned} a &\to a = a', \\ a_D &\to a_D + a - a_0 = a_D + a - \frac{m_f}{\sqrt{2}} = a'_D, \end{aligned} \tag{3.41}$$

showing that the pair (a_D, a) is not simply transformed by SL(2, Z) but they also pick up the additional constant m_f. In the strong coupling region, the s_f charges contribute to a also. Under the transformation (3.41), we obviously have

$$\begin{aligned} s'_f &= n_m + s_f, \\ n'_m &= n_m, \\ n'_e &= n_e - n_m, \end{aligned} \tag{3.42}$$

showing that the duality in the theory with nonzero bare mass is really an electric–magnetic s_f duality.

In $N_f = 1$ case, the moduli space of vacua is homomorphic to the compactified complex u-plane, and the SU(2) gauge group is broken down to U(1) on the coulomb branch which exhibits Z_3-symmetry. The Higg's branch is absent in this case. For vanishing bare masses, the torus is

$$y^2 = x^3 - ux^2 - \Lambda^2/64, \tag{3.43}$$

with $da_{\mathrm{D}}^{(1)}/du$ and $da^{(1)}/du$ as its period integrals. Here the weak coupling region R_{W} contains all dyons $(n, 1)$ as well as elementary quarks and W-bosons while the strong coupling region R_{S} contains the monopoles $\pm(0, 1)$ and dyons $\pm(-1, 1)$ and $\pm(2, -1)$. For $N_{\mathrm{f}} = 2$ case, in addition to Coulomb branch, the moduli space also has a Higg's branch on which the gauge group is completely broken. The strong coupling region R_{S} in this case contains only one Z_2-pair incorporating monopoles $\pm(0, 1)$ and the dyons $\pm(\pm 1, 1)$. For $N_{\mathrm{f}} = 3$ case, the equation of the torus for vanishing bare mass is

$$y^2 = (x - u)(x^2 - x + u). \tag{3.44}$$

Here the strong coupling region contains the dyons $\pm(-1, 2)$ and $\pm(1, -1)$ and the monopoles $\pm(0, 1)$ while the weak coupling region contains the dyons $(n, 1)$ and $(2n + 1, 2)$ for all integer n.

4. BPS SPECTRUM OF DYONIC STATES IN STRING THEORY

In general a $(n_{\mathrm{e}}, n_{\mathrm{m}})$ hypermultiplet dyon state in the probe theory corresponds to a $(n_{\mathrm{e}}, n_{\mathrm{m}})$ string stretched between the D_3-brane probe and a 7-brane. Such state exists as long as there exists a path P along which the total monodromy transforms the $(n_{\mathrm{e}}, n_{\mathrm{m}})$ charge of the string to the charge of the 7-brane. This state is BPS state if the path corresponds to a $(n_{\mathrm{e}}, n_{\mathrm{m}})$ geodesic and it minimizes the mass given by

$$M = \int_{\mathrm{P}} T_{(n_{\mathrm{e}}, n_{\mathrm{m}})}\, ds, \tag{4.1}$$

where

$$T_{(n_{\mathrm{e}}, n_{\mathrm{m}})} = \frac{1}{\sqrt{\operatorname{Im} \tau}} |n_{\mathrm{e}} - n_{\mathrm{m}} \tau|, \tag{4.2}$$

is the tension of $(n_{\mathrm{e}}, n_{\mathrm{m}})$ string with τ as string coupling which corresponds to effective coupling constant defined by Eq. (3.4) and (3.21). The $(n_{\mathrm{e}}, n_{\mathrm{m}})$ charges of the string may change along the path P according to transformations (3.7) and (3.9) if this path does not pass through any of the singularities. In case of singularities at $u = \Lambda^2$ and $u = -\Lambda^2$, these charges will change according to transformations (3.35b) and (3.36a) respectively. For $N_{\mathrm{f}} = 0$, the metric ds^2 is given by (Bergman and Fayyazuddin, 1998)

$$ds^2 \operatorname{Im}(\tau) \left| \frac{\eta^2(\tau)}{\sqrt{2\Lambda^{2/3}}} [(Z - Z_1)(Z - Z_2)]^{-1/2}\, dz \right|^2, \tag{4.3}$$

where $Z_1 = 4\Lambda^2$ and $Z_2 = -4\Lambda^2$ are the positions of (0, 1) and (1, 1) 7-branes, respectively.

Using relations (3.4) and (3.21), in Eq. (4.2), we have

$$T_{(n_{\mathrm{e}}, n_{\mathrm{m}})}\, ds = |n_{\mathrm{e}}\, da - n_{\mathrm{m}}\, da_{\mathrm{D}}|, \tag{4.4}$$

which implies that the geodesic for $(n_{\rm e}, n_{\rm m})$ string is

$$n_{\rm e}\frac{da}{dt} + n_{\rm m}\frac{da_{\rm D}}{dt} = (n_{\rm e} - n_{\rm m}\tau)\frac{da}{dt} = C, \tag{4.5}$$

where C is constant. For topologically trivial path which does not go around 7-branes, the solution of this equation is

$$n_{\rm e}a\{Z(t)\} - n_{\rm m}a_{\rm D}\{Z(t)\} = C(t-1), \tag{4.6}$$

where $Z(0) = u$ is the position of the D_3-brane and $Z(1)$ is the position of $(n_{\rm e}, n_{\rm m})$ 7-brane. The mass can then be calculated as

$$M = \sqrt{2}|C| = \sqrt{2}|n_{\rm e}a(u) - n_{\rm m}a_{\rm D}(u)|,$$

which is identical to Eq. (3.12). The only states which correspond to topological trivial geodesic are (0, 1), (1, 1) hypermultiplets, as the charges of 7-branes. This is consistent with the result, obtained in the previous section, that the BPS spectrum in $R_{\rm S}$ region consists of only (0, 1) and $(1, \pm1)$ massless states.

W-boson i.e., (1, 0) state, corresponds to a pair of fundamental strings beginning on the D_3-brane and ending on the r^7-plane. Quantum corrections deform this background in to two separated 7-branes with charges (0, 1), and (2, 1) and at the same time the W-boson state is deformed accordingly. Most likely possibility is that it is deformed into four-pronged strings with external prongs (1, 0), (1, 0), (0, 1), and (2, 1) and an internal prong (1, 1) such that the first two external prongs end on the D_3-brane, and the last two end on the corresponding 7-branes. Supersymmetry requires that the two (1, 0) strings be parallel at the location of the D_3-brane. This condition is satisfied for the four-pronged string configuration if the internal (1, 1) string shrinks to zero length. Denoting the locations of (0, 1) 7-brane, (2, 1) 7-brane and the D_3-brane by Z_1, Z_2, and Z_3 respectively, and the location of the two coincident junction points by u, we have following three geodesic

$$\begin{aligned} P_1:&\quad a_{\rm D}\{Z(t_1)\} = c_1t_1 + a_{\rm D}(u),\\ P_2:&\quad -2a\{Z(t_2)\} - a_{\rm D}\{Z(t_2)\} = c_2t_2 - 2a(u) - a_{\rm D}(u),\\ P_3:&\quad a\{Z(t_3)\} = c_3t_3 + a(u), \end{aligned} \tag{4.7}$$

where t_1, t_2, and t_3 respectively parameterize the geodesics for (0, 1), (2, 1), and (1, 0) strings and $t_i = 0$ corresponds to the position of junction while $t_i = 1$ corresponds to the position of the brane. In Eqs. (4.7) we have set up

$$\begin{aligned} C_1 &= -a_{\rm D}(u),\\ C_2 &= 2a(u) + a_{\rm D}(u)\\ C_3 &= a(Z_3) - a(u), \end{aligned} \tag{4.8}$$

and

$$n_e a - n_m a_D = 0,$$

at the location of (n_e, n_m) 7-brane. Because of the orientation of (1, 0) string at the flat space, the supersymmetry requires that

$$\frac{C_1}{|C_1|} = \frac{C_2}{|C_2|} = \frac{C_3}{|C_3|}. \tag{4.9}$$

Using Eqs. (4.8) and the first equality of (4.9), we have

$$\mathrm{Im}\frac{a_D(u)}{a(u)} = 0, \tag{4.10}$$

and

$$\frac{a_D(u)}{a(u)} > -2, \tag{4.10a}$$

which requires u to be on the curve of marginal stability defined by Eq. (3.32).

Equations (4.8) and the second equality of (4.9) imply that

$$\mathrm{Im}\frac{a(Z_3)}{a(u)} = 0, \tag{4.11}$$

and

$$\frac{a(Z_3)}{a(u)} > 1, \tag{4.11a}$$

showing that the point Z_3 lies in weak coupling region R_W. In other words, the state corresponding to the four-pronged string is BPS-state only when D_3-brane is in R_W region. At $Z_3 = u$, the D_3-brane is exactly on the curve of marginal stability and then the (1, 0) prongs will degenerate and the remaining (0, 1) and (2, 1) strings will separate along the D_3-brane.

The total mass is the sum of four individual prong masses;

$$\begin{aligned} M &= |C_1| + |C_2| + 2|C_3| \\ &= \sqrt{2}|a(Z_3)|, \end{aligned}$$

which is the BPS mass of the W-boson.

In addition to W-bosons and the two hypermultiplets (0, 1) and (1, 1), the BPS spectrum in R_W-region also includes hypermultiplets carrying charges $(n, 1)$ with $n > 1$. These states can be obtained from (0, 1) and (1, 1) states by applying the monodromy transformations (3.35b) and (3.36a).

5. DISCUSSION

The Eq. (2.13) describes the equation of mass-deformed $N = 4$ theory which may be written as Eq. (2.15) by changing the variables given by Eq. (2.14). This equation further reduces to the form given by Eq. (2.18) under conditions described by Eqs. (2.16) and (2.17). Equation (2.18) is the curve of the expected low energy pure gauge $N = 2$ theory in the $N = 4$ convention. In the absence of hypermultiplets, the explicit forms of a and $a_{\rm D}$ given by Eq. (3.33) in terms of hypergeometric functions show that the branch points $u = \pm 1$ and $u = \infty$ are the singularities of the moduli space, for which the monodromy matrices given by Eqs. (3.35a), (3.36), and (3.37) generate the transformations of parameter u of quantum moduli space to another point at which the BPS state become massless. It shows that the singular points of moduli space are associated with extra massless particles. Monodromy M_1, given by Eq. (3.35a) arises from the massless monopole of charge $(n_{\rm e}, n_{\rm m}) = (0, 1)$ and the corresponding transformations (3.35b) transforms the electron (0, 1) near $u = 1$ to a dyon (1, 2). The monodromy (3.36) arises from the vanishing mass of dyon $(-1, 1)$ and the corresponding transformations (3.36b) transform an electron to a dyon $(-1, 2)$ and a monopole (0, 1) to a dyon $(-2, 3)$. Under the monodromy (3.37) at infinity, the charge transforms according to Eq. (3.38) showing that at this singularity a monopole (0, 1) gains the electric charge and becomes a dyon $(-2, -1)$. This monodromy at infinity implies that there must be an additional singularity somewhere is u-plane. For instance, a monopole becomes massless at a point where $a_{\rm D} = 0$ while $a \neq 0$ and a dyon with charge $(-1, 1)$ becomes massless if $a + a_{\rm D} = 0$ while $a, a_{\rm D} = 0$.

The matrices, given by Eqs. (3.35a), (3.36), and (3.37) are the three generators of monodromies at the points $u = 1, -1$, and ∞ respectively, where the complex u-plane is punctured. The pair $(a_{\rm D}, a)$ forms a holomorphic section of SL(2, Z) bundle over the punctured u-plane. These matrices generate the subgroup $\Gamma(2) \in SL(2, Z)$ of 2×2 matrices congruent to unit matrix modulo 2. In fact M_∞ and M_1 do penetrate $\Gamma(2)$. The u-plane punctured at $1, -1$, and ∞ can be thought of as the quotient of the upper plane $H/\Gamma(2)$. The family of curves parameterized by $H/\Gamma(2)$ can explicitly be described by Eq. (3.15), for $\Lambda = 1$, which is symmetrical under the mapping

$$u \to -u, x \to -x, y \to \pm iy.$$

These transformations generate Z_4 symmetry, but only its Z_2 quotient acts on u-plane. In looping around $u = 1$ and $u = -1$, the pair (a_η, a) are transformed by the monodromies M_1 and M_{-1} and the charges $(n_{\rm e}, n_{\rm m})$ are transformed accordingly. But the spectrum of BPS saturated states is not $\Gamma(2)$ invariant. This lack of duality in the spectrum can be resolved if the curve, on which $a_{\rm D}/a$ is real, looks something like $|u| = 1$. Then one can avoid the jumping phenomena only if one stays in the region $u > 1$. The only monodromy that can be seen in that region is M_∞, under which the spectrum of BPS saturated states is invariant. Whatever particles become massless at $u = \pm 1$ must evolve continuously from the BPS—saturated states that can be seen semiclassically near infinity. It follows from the foregoing analysis that the entire BPS spectrum in strong coupling as well as weak coupling regions can be accounted for in D_3-brane probe picture, either as open strings or topologically geodesics between the D_3-brane and a single 7-brane, or as multipronged strings connecting the D_3-brane to both 7-branes. The former correspond to hypermultiplets carrying charges (0, 1) and (1, 1) and exist everywhere in the moduli space. The latter correspond to the W-bosons and hypermultiplets carrying charges $(n, 1)$ for $(n > 1)$, and exist only in the weak coupling region (i.e., out side the curve of marginal stability). It has also been shown here that for W-boson the multipronged string is the unique BPS representation and there are no topologically nontrivial geodesics beginning and ending on the D_3-brane.

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